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Complexities and Their Applications to Characterization of Chaos[2]

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The concept of complexity in Information Dynamics is discussed. The chaos degree defined by the complexities is applied to examine chaotic behavior of logistic map.

1. INTRODUCTION

There are several tools to describe chaotic aspects of natural or nonnatural phenomena such as entropy. The concept of complexity is one such tool. In 1991 the author proposed Information Dynamics (ID, for short) to synthesize the dynamics of state change and the complexity of a system. In this paper, I briefly review the concept of ID and discuss some applications of the entropic complexities in ID to the characterization of chaos.

2. INFORMATION DYNAMICS

Information Dynamics is an attempt to provide a new view for the study of chaotic behavior of systems (Ohya, 1995).

Let $(\mathcal{A}, \mathfrak{S}, \alpha(G))$ be an input (or initial) system and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{G}))$ be an output (or final) system. Here $\mathcal{A}$ is the set of all objects to be observed and $\mathfrak{S}$ is the set of all means of getting the observed value, $\alpha(G)$ is a certain evolution of the system. Often we have $\mathcal{A} = \overline{\mathcal{A}}$, $\mathfrak{S} = \overline{\mathfrak{S}}$, $\alpha = \overline{\alpha}$.

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[2] Dedicated to Professor Gérard G. Emch on his 60th birthday.

Therefore we claim

Giving a mathematical structure to input and output triples

$\equiv$ Having a theory

For instance, when $\mathscr{A}$ is the set $M(\Omega)$ of all measurable functions on a measurable space $(\Omega, \mathscr{F})$ and $\mathfrak{S}(\mathscr{A})$ is the set $P(\Omega)$ of all probability measures on Ω, we have usual probability theory, by which the classical dynamical system is described. When $\mathscr{A} = B(\mathscr{H})$, the set of all bounded linear operators on a Hilbert space $\mathscr{H}$, and $\mathfrak{S}(\mathscr{A}) = \mathfrak{S}(\mathscr{H})$, the set of all density operators on $\mathscr{H}$, we have a quantum dynamical system.

The dynamics of state change is described by a channel $\Lambda^*: \mathfrak{S} \to \overline{\mathfrak{S}}$ (sometimes $\mathfrak{S} \to \mathfrak{S}$). The fundamental point of ID is that there exist two complexities in ID itself.

Let $(\mathscr{A}_i, \mathfrak{S}_i, \alpha^t(G_t))$ be the total system of $(\mathscr{A}, \mathfrak{S}, \alpha)$ and $(\overline{\mathscr{A}}, \overline{\mathfrak{S}}, \overline{\alpha})$, and $\mathscr{S}$ be a subset of $\mathfrak{S}$ from which we measure the observables and we call this subset a reference system [e.g., $\mathscr{S} = I(\alpha)$, the set of all invariant elements of α].

$G^{\mathscr{S}}(\varphi)$ is the complexity of a state φ measured from $\mathscr{S}$ and $T^{\mathscr{S}}(\varphi; \Lambda^*)$ is the transmitted complexity associated with the state change $\varphi \to \Lambda^*\varphi$, which satisfy the following properties:

(i) For any $\varphi \in \mathscr{S} \subset \mathfrak{C}$,

$$C^{\mathscr{S}}(\varphi) \geq 0, \qquad T^{\mathscr{S}}(\varphi; \Lambda^*) \geq 0$$

(ii) For any orthogonal bijection $\bar{j} : ex\,\mathscr{S} \to ex\,\mathscr{S}$ (the set of all extreme points in $\mathscr{S}$),

$$C^{j(\mathscr{S})}(j(\varphi)) = C^{\mathscr{S}}(\varphi)$$
$$T^{j(\mathscr{S})}(j(\varphi); \Lambda^*) \doteq T^{\mathscr{S}}(\varphi; \Lambda^*)$$

(iii) For $\Phi \equiv \varphi \otimes \psi \in \mathscr{S}_t \subset \mathfrak{S}_t$,

$$C^{\mathscr{S}_t}(\Phi) = C^{\mathscr{S}}(\varphi) + C^{\overline{\mathscr{S}}}(\psi)$$

(iv) For any state φ and a channel Λ^*,

$$0 \leq T^{\mathscr{S}}(\varphi; \Lambda^*) \leq C^{\mathscr{S}}(\varphi)$$

(v) For the identity map id from $\mathfrak{S}$ to GS,

$$T^{\mathscr{S}}(\varphi; id) = C^{\mathscr{S}}(\varphi)$$

Instead of (iii), when "(iii′) $\Phi \in \mathscr{S}_t \subset \mathfrak{S}_t$, put $\varphi \equiv \Phi \upharpoonright_{\mathscr{A}}$ (i.e., the restriction of Φ to $\mathscr{A}$), $\psi \equiv \Phi \upharpoonright_{\mathscr{A}}$, $C^{\mathscr{S}_t}(\Phi) \leq C^{\mathscr{S}}(\varphi) + C^{\mathscr{S}}(\psi)$" is satisfied,

C and T are called a pair of strong complexity. Therefore ID can be considered as follows.

Definition 1. Information Dynamics (ID) is defined by

$$(\mathcal{A}, \mathfrak{S}, \alpha(G); \overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{G}); \Lambda^*; C^{\mathcal{S}}(\varphi), T^{\mathcal{S}}, (\varphi; \Lambda^*))$$

and some relations R among them.

Thus, in the framework of ID, we have to:
(i) Determine mathematically

$$\mathcal{A}, \mathfrak{S}, \alpha(G); \overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{G})$$

(ii) Choose Λ^* and R.
(iii) Define $C^S(\varphi)$, T^S $(\varphi; \Lambda^*)$.

Information Dynamics can be applied to the study of chaos in the following ways:

(a) ψ is more chaotic than φ as seen from the reference system $\mathcal{S}$ if $C^{\mathcal{S}}(\psi) \geq C^{\mathcal{S}}(\varphi)$.

(b) When φ changes to $\Lambda^*\varphi$, a *degree of chaos* associated to this state change is given by

$$D^{\mathcal{S}}(\varphi; \Lambda^*) = C^{\overline{\mathcal{S}}}(\Lambda^*\varphi) - T^{\mathcal{S}}(\varphi; \Lambda^*)$$

In ID, several different topics can be treated from a common standpoint (Matsuoka and Ohya, 1995; Ohya, 1991a, n.d.-a, c; Ohya and Watanabe, 1993). Although there exist several complexities (Ohya, 1997), one of the most fundamental pairs of C and T in quantum system is the von Neumann entropy and the mutual entropy, whose C and T are modified to formulate the entropic complexities such as ε-entropy (ε-entropic complexity) (Ohya, 1989, 1991b, 1995) and Kolmogorov–Sinai type dynamical entropy (entropic complexity) (Accardi *et al.*, 1996; Muraki and Ohya, 1996).

In this paper, we discuss some applications of entropic complexities to the study of chaos.

3. CHANNEL

The concept of channel or channeling transformation is fundamental in ID and it is a convenient mathematical tool to treat several physical dynamics in a unified way (Ohya, 1981).

In classical systems, an input (or initial) system is described by the set of all random variables $\mathcal{A} = M(\Omega)$ and its state space $\mathfrak{S} = P(\Omega)$, and an output (or final) system by $M(\overline{\Omega})$ and $P(\overline{\Omega})$.

A quantum system is described on a Hilbert space $\mathcal{H}$. That is, an input $\mathcal{A}$ is the set $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, and $\mathfrak{S}$ is the set $T(\mathcal{H})$ of all density operators on $\mathcal{H}$. An output system is $\bar{\mathcal{A}} = B(\bar{\mathcal{H}})$ and $\bar{\mathfrak{S}} = T(\bar{\mathcal{H}})$. A more general quantum system is described by a C*-algebra and its space, but this general frame is not used in this paper.

In any case, a channel is a mapping from $\mathfrak{S}(P(\Omega))$ or $T(\mathcal{H})$, resp.) to $\bar{\mathfrak{S}}$ ($P(\bar{\Omega})$ or $T(\bar{\mathcal{H}})$, *resp.*). Almost all physical transformations are described by this mapping.

Definition 2. Let Λ^* be a channel from $\mathfrak{S}$ to $\bar{\mathfrak{S}}$.

(1) Λ^* is linear if $\Lambda^*(\lambda\varphi + (1-\lambda)\psi) = \lambda\Lambda^*\varphi + (1-\lambda)\Lambda^*\psi$ holds for all $\varphi, \psi \in \mathfrak{S}$ and any $\lambda \in [0, 1]$.

(2) Λ^* is completely positive (C. P.) if Λ^* is linear and its dual Λ: $\bar{\mathcal{A}} \rightarrow \mathcal{A}$ satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(\bar{A}_i^* \bar{A}_j) A_j \geq 0$$

for any $n \in \mathbf{N}$ and any $\{\bar{\mathcal{A}}_i\} \subset \bar{\mathcal{A}}$, $\{A_i\} \subset \mathcal{A}$.

Most channels appearing in physical processes are C.P. channels. We here list a few examples of such channels (Ohya, 1989). Take a density operator ρ as an input (initial) state.

(1) *Time evolution*: Let $\{U_t\,;\, t \in \mathbf{R}^+\}$ be one-parameter group or semi-group on $\mathcal{H}$. We have

$$\rho \rightarrow \Lambda_t^*\rho = U_t \rho U_t^*$$

(2) *Quantum measurement*: When a measuring apparatus is described by a positive operator-valued measure $\{Q_n\}$ and the measurement is carried out in a state ρ, the state ρ changes to a state $\Lambda^*\rho$ by this measurement such that

$$\rho \rightarrow \Lambda^*\rho = \sum_n Q_n^{1/2} \rho Q_n^{1/2}$$

(3) *Reduction*: If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and σ, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between two systems is given by

$$\theta_t \equiv U_t(\rho \otimes \sigma) U_t^*$$

where $U_t = \exp(-itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho \rightarrow \Lambda_t^*\rho \equiv \mathrm{tr}_{\mathcal{K}}\theta_t$$

4. QUANTUM ENTROPY AS COMPLEXITY

The concept of entropy was introduced and developed to study the following topics: irreversible behavior, symmetry breaking, amount of information transmission, chaotic properties of states, etc. Here we review quantum entropies as an example of our complexities C and T.

A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropy of a state ρ was introduced by von Neumann (1932; Ohya and Petz, 1993) as

$$S(\rho) = -\mathrm{tr}\, \rho \log \rho$$

If $\rho = \Sigma_k p_k E_k$ is the Schatten decomposition (i.e., p_k is the eigenvalue of ρ and E_k is the one-dimensional projection associated with p_k, this decomposition is not unique unless every eigenvalue is nondegenerate of ρ, then

$$S(\rho) = -\sum_k p_k \log p_k$$

because $\{p_k\}$ is a probability distribution. Therefore the von Neumann entropy contains the Shannon entropy as a special case.

For two states $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$, the relative entropy (Umegaki, 1962) is defined by

$$S(\rho, \sigma) = \begin{cases} \mathrm{tr}\, \rho(\log \rho - \log \sigma), & \rho << \sigma \\ +\infty, & \text{otherwise} \end{cases}$$

where $\rho << \sigma$ means that $\mathrm{tr}\, \sigma A = 0 \Rightarrow \mathrm{tr}\, \rho A = 0$ for any $A \geq 0$.

Let $\Lambda^*\colon \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H})$ be a channel and define the compound state by

$$\theta_E = \sum_k p_k E_k \otimes \Lambda^* E_k$$

which expresses the correlation between the initial state ρ and the final state $\Lambda^*\rho$ (Ohya, 1983a, b). The mutual entropy (Ohya, 1983a) for a state $\rho \in \mathfrak{S}(\mathcal{H})$ and a channel Λ^* is given by

$$\begin{aligned} I(\rho; \Lambda^*) &= \sup\{S(\theta_E, \rho \otimes \Lambda^*\rho);\ E = \{E_k\}\} \\ &= \sup\left\{\sum_k p_k S(\Lambda^* E_k, \Lambda^*\rho);\ E = \{E_k\}\right\} \end{aligned}$$

where the supremum is taken over all Schatten decompositions. The above entropy and mutual entropy become a pair of our two complexities according to the following facts:

(1) The fundamental inequality of Shannon type (Shannon 1948; Ohya, 1983a)

$$0 \le I(\rho; \Lambda^*) \le \min\{S(\rho), S(\Lambda^*\rho)\}$$

because of $S(\Lambda^* E_k, \Lambda^*\rho) = S(\Lambda^*\rho) - \Sigma_k p_k S(\Lambda^* E_k) \le S(\Lambda^*\rho)$ and the monotonicity (Uhlmann, 1977; Ohya and Petz, 1993) of the relative entropy: $S(\Lambda^* E_k, \Lambda^*\rho) \le S(E_k, \rho)$.

(2) $I(\rho; id) = S(\rho)$, which is proved as follows:

$$I(\rho; \text{id}) = \sup\left\{\sum_k p_k S(E_k, \rho);\ E = \{E_k\}\right\}$$

$$= \sup\left\{S(\rho) - \sum_k p_k S(E_k) E = \{E_k\}\right\} = S(\rho)$$

because of $S(E_k) = 0$.

In Shannon's communication theory in classical systems, ρ is a probability distribution $p = (p_k)$ and Λ^* is a transition probability (t_{ij}), so that the Schatten decomposition of ρ is unique and the compound state of ρ and its output $\overline{\rho}$ $[\equiv \overline{p} = (\overline{p}_i)]$ is the joint distribution $r = (r_{ij})$ with $r_{ij} \equiv t_{ij} p_j$. Then the above complexities C and T become the Shannon entropy and mutual entropy, respectively,

$$S(p) = -\sum_k p_k \log p_k$$

$$I(p; \Lambda^*) = \sum_{i,j} r_{ij} \log \frac{r_{ij}}{p_j \overline{p}_i}$$

We can construct several other types of entropic complexities and they are used to define the quantum dynamical entropy (Muraki and Ohya, 1996; Ohya, n.d.-a; Accardi *et al.*, 1996), which is one of fundamental tools to describe chaotic aspects of a dynamical system (Billingsley, 1965; Benatti, 1993; Connes *et al.*, 1987; Connes and Stormer, 1975; Emch, 1975). For instance, one pair of the complexities is

$$T(\rho; \Lambda^*) = \sup\left\{\sum_k p_k S(\Lambda^*\rho_k, \Lambda^*\rho);\ \rho = \sum_k p_k \rho_k\right\}, \qquad C(\rho) = T(\rho; id)$$

where $\rho = \Sigma_k p_k \rho_k$ is a finite decomposition of ρ and the supremum is taken over all such finite decompositions.

The mutual entropy given above contains other definitions of the mutual information (Holevo, 1973; Ingarden, 1976; Levitin, 1991). Moreover it is not only a fundamental quantity to study quantum communication processes such as the capacity of a quantum channel (Ohya, n.d.-b; Ohya *et al.*, 1997), but also can be used to study irreversible processes (Ohya, 1989).

5. CHAOS DEGREE

We apply the chaos degree in ID to a deterministic dynamical system and discuss its usefulness. The degree of chaos for a state ρ (density operator or probability distribution) and a channel Λ^* is defined as

$$D(\rho;\, \Lambda^*) \equiv C(\Lambda^*\rho) - T(\rho;\, \Lambda^*)$$
$$\equiv S(\Lambda^*\rho) - I(\rho;\, \Lambda^*)$$

We shall see how this degree works to describe the chaotic aspects of a logistic map. The logistic map is given by the following equation:

$$x_{n+1} = f_a(x_n) = ax_n(1 - x_n), \qquad x_n \in [0, 1], \qquad 0 \le a \le 4 \quad (5.1)$$

The solution of this equation bifurcates as shown in Fig. 1.

The Lyapunov exponent of this map has been calculated by Shaw (Shaw, 1981) (Fig. 2). The Lyapunov exponent λ is defined as

$$\lambda_n = \frac{1}{n}\sum_{k=1}^{n} \log \left|\frac{df_a}{dx}(x_k)\right|, \qquad \lambda = \lim_{n\to\infty} \lambda_n$$

Figure 2 is the result of computing λ_n for 1000 a's from 3.0 to 4.0 with

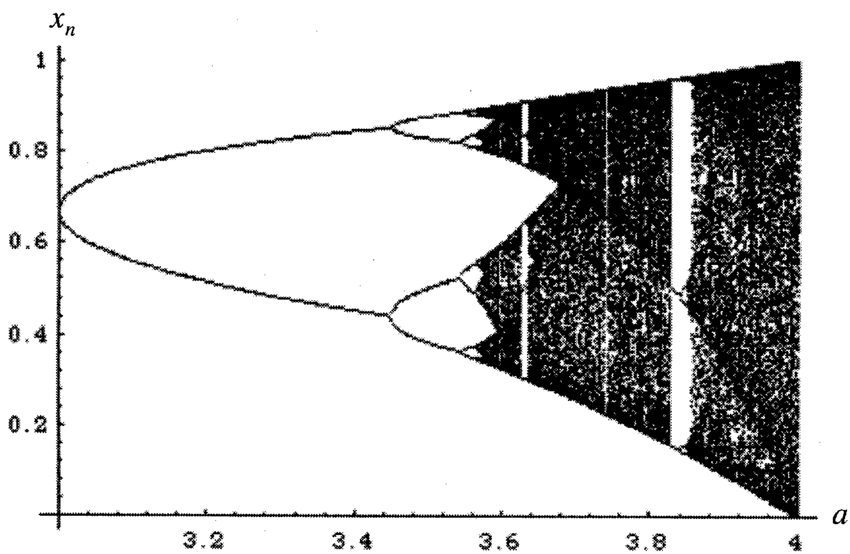

Fig. 1. Bifurcation diagram of logistic map.

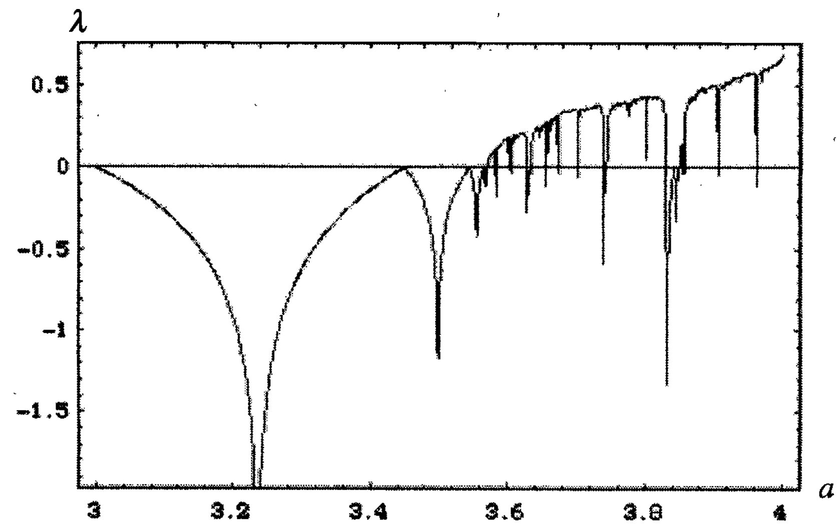

Fig. 2. Lyapunov exponent of logistic map.

$n = 100{,}000$ steps. A positive exponent means that the trajectory is very sensitive to the initial value and is called chaotic; a negative exponent means that the trajectory is stable.

If a logistic map f_a does not have a stable and periodic trajectory, then there exists an ergodic probability measure μ on the Borel set of [0,1], absolutely continuous with respect to the Lebesgue measure (Misiurewicz, 1981).

Take a finite partition $\{A_k\}$ of $I = [0,1]$ such as

$$I = \bigcup_{k=1}^{m} A_k \qquad (A_i \cap A_j = \emptyset,\ i \neq j)$$

Let $|Q|$ be the number of the elements in a set Q and $p^{(n)} \equiv (p_k^{(n)})$ be the probability distribution of the trajectory up the nth step, that is, how many x_j ($j = 1, \ldots, n-1$) are in A_k:

$$p_k^{(n)} = \frac{|\{j;\ x_j \in A_k,\ 1 \leq j \leq n\}|}{n}, \qquad n \geq 1$$

This probability distribution is obviously from the difference equation of (5.1), hence it depends on the initial value x_1 and f_a. The channel Λ^* is a map given by $p^{(n+1)} = \Lambda^* p^{(n)}$. It can be shown (Misiurewicz, 1981) that the $n \to \infty$ limit of $p_k^{(n)}$ exists and is equal to $\mu(A_k)$. Further, the joint distribution $r^{(n,n+1)} = (r_{ij}^{(n,n+1)})$ for a sufficiently large n is approximated as

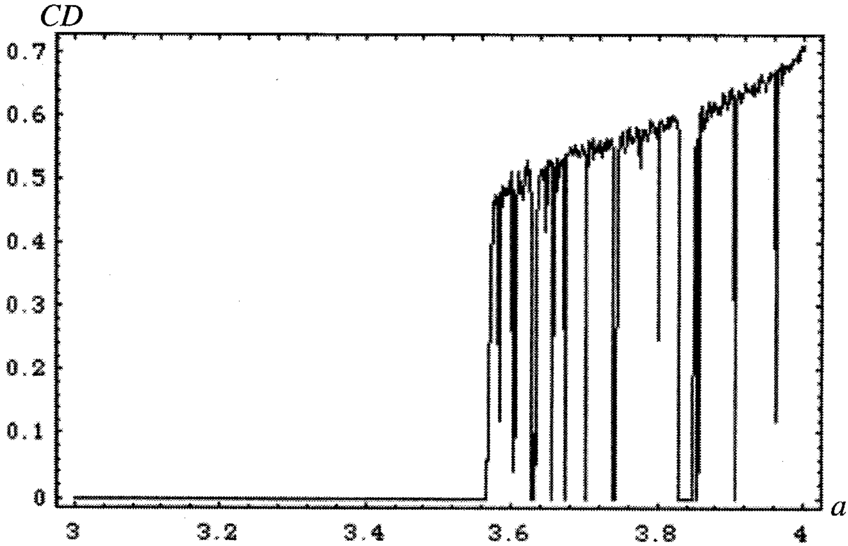

Fig. 3. Chaos degree.

$$r_{ij}^{(n,n+1)} = \frac{|\{k;\ (x_k, x_{k+1}) \in A_i \times A_j,\ 1 \leq k \leq n\}|}{n}$$

Then the chaos degree (CD) is

$$\begin{aligned} D(p^{(n)}; \Lambda^*) &\equiv C(\Lambda^* p^{(n)}) - T(p^{(n)}; \Lambda^*) \\ &= S(\Lambda^* p^{(n)}) - I(p^{(n)}; \Lambda^*) \\ &= S(p^{(n+1)}) - I(p^{(n)}; \Lambda^*) \end{aligned}$$

For a computer simulation, we take 1000 a's from 3.0 to 4.0 and

$$A_i = \left[\frac{i}{2000}, \frac{i+1}{2000}\right], \qquad i = 0, \ldots, 1999$$

$$n = 100{,}000$$

The choice of these quantities does not alter the results so much if we take large n.

The result is shown in Fig. 3, from which we conclude that our chaos degree describes the chaotic aspects of the logistic map, namely,

$$D > 0 \Leftrightarrow \text{chaotic}$$

$$D = 0 \Leftrightarrow \text{non-chaotic}$$

Although the Lyapunov exponent becomes negative, sometimes-∞, our degree is always nonnegative, so that it might be useful to make the chaotic domain clear-cut. More rigorous study of the chaos degree and its use for other dynamical channels including quantum systems is now in progress.

REFERENCES

Accardi, L., Ohya, M., and Watanabe, N. (1996). Note on quantum dynamical entropy, *Reports on Mathematical Physics,* **38**, 457–469.

Benatti, F. (1993). *Deterministic Chaos in Infinite Quantum Systems*, Springer-Verlag, Berlin.

Billingsley, P. (1965). *Ergodic Theory and Information*, Wiley, New York.

Connes, A., and Størmer, E. (1975). Entropy for automorphisms of II_1 von Neumann algebras, *Acta Mathematica,* **134**, 289–306.

Connes, A., Narnhofer, H., and Thirring, W. (1987). Dynamical entropy of C*-algebras and von Neumann algebras, *Communications in Mathematical Physics,* **112**, 691–719.

Emch, G. G. (1974). Positivity of the K-entropy on non-abelian K-flows, *Zeitschrift für Wahrscheinlichkeits theorie Verwandte Gebiete,* **29**, 241–252.

Holevo, A. S. (1973). Some estimates for the amount of information transmittable by a quantum communication channel, *Problemy Peredachi Informacii,* **9**, 3–11 [in Russian].

Ingarden, R. S. (1976). Quantum information theory, *Reports on Mathematical Physics,* **10**, 43–73.

Ingarden, R. S., Kossakowski, A., and Ohya, M. (1997). *Information Dynamics and Open System*, Kluwer Academic Publishers.

Levitin, L. B. (1991). Physical information theory for 30 years: basic concepts and results, in *Springer Lecture Notes in Physics*, Vol. **378**, pp. 101–110.

Matsuoka, T., and Ohya, M. (1995). Fractal dimensions of states and its application to Ising model *Reports on Mathematical Physics,* **36**, 365–379.

Misiurewicz, M. (1981). Absolutely continuous measures for certain maps of interval, *Publications Mathematiques IHES,* **53**, 17–51.

Muraki, N., and Ohya, M. (1996). Entropy functionals of Kolmogorov Sinai type and their limit theorems, *Letters in Mathematical Physics,* **36**, 327–335.

Ohya, M. (1981). Quantum ergodic channels in operator algebras, *Journal of Mathematical Analysis and Applications,* **84**, 318–327.

Ohya, M. (1983a). On compound state and mutual information in quantum information theory, *IEEE Transaction of Information Theory,* **29**, 770–774.

Ohya, M. (1983b). Note on quantum probability, *Lettere al Nuovo Cimento,* **38**, 402–406.

Ohya, M. (1989). Some aspects of quantum information theory and their applications to irreversible processes, *Reports on Mathematical Physics,* **27**, 19–47.

Ohya, M. (1991a). Information dynamics and its application to optical communication processes, *Springer Lecture Notes in Physics*, Vol. **378**, pp. 81–92.

Ohya, M. (1991b). Fractal dimension of states, *Quantum Probability and Related Topics,* **6**, 359–369.

Ohya, M. (1995). State change, complexity and fractal in quantum systems, *Quantum Communications and Measurement,* **2**, 309–320.

Ohya, M. (n.d.-a). Foundation of entropy, complexity and fractal in quantum systems, in *International Congress of Probability toward 2000*, to appear.

Ohya, M. (n.d.-b). Fundamentals of quantum mutual entropy and capacity, submitted.

Ohya, M. (n.d.-c). Applications of information dynamics to genome sequences, SUT preprint.

Ohya, M. (1997). Complexity and fractal dimensions for quantum states, *Open Systems and Information Dynamics,* **4**, 141–157.

Ohya, M., and Petz, D. (1993). *Quantum Entropy and Its Use*, Springer-Verlag, Berlin.

Ohya M., and Watanabe N. (1993). Information dynamics and its application to Gaussian communication process, *Maximum Entropy and Bayesian Method,* **12**, 195–203.

Ohya, M., Petz, D., and Watanabe, N. (1997). On capacities of quantum channels, *Probability and Mathematical Statistics,* **17**, 179–196.

Shannon, C. E. (1948). Mathematical theory of communication, *Bell System Technical Journal,* **27**, 379–423.

Shaw, R. (1981). Strange attractors, chaotic behavior and information flow, *Zeitschrift für Naturforschung,* **36a**, 80–112.

Uhlmann, A. (1977). Relative entropy and the Wigner–Yanase–Dyson–Lieb concavity in interpolation theory, *Communication in Mathematical Physics,* **54**, 21–32.

Umegaki, H. (1962). Conditional expectations in an operator algebra IV (entropy and information), *Kodai Mathematical Seminar Reports,* **14**, 59–85.

Von Neumann, J. (1932). Mathematischen Grundlagen *Quantenmechanik* , Springer, Berlin.